Example 8: Consider the vectors

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \quad \mathbf{v}_4 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \tag{5}$$

Since $\underline{\mathbf{v}_3 = \mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_4}$ we have that $\underline{\mathbf{v}_3}$ is in $\underline{\mathrm{span}(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_4)}$ and hence by theorem 3

$$\mathrm{span}(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4) = \mathrm{span}(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_4) \tag{6}$$

Example 9: For any vectors $\mathbf{v}_1, \ldots, \mathbf{v}_{i-1}, \ldots, \mathbf{v}_{i+1}, \ldots, \mathbf{v}_n$ in $\mathbb{R}^m$ we have

$$\mathrm{span}\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \cancel{\begin{bmatrix} 0 \\ 0 \end{bmatrix}}, \begin{bmatrix} 1 \\ 2 \end{bmatrix}\right) = \mathrm{span}\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix}\right)$$

$$\mathrm{span}(\mathbf{v}_1, \ldots, \mathbf{v}_{i-1}, \mathbf{0}, \mathbf{v}_{i+1}, \ldots, \mathbf{v}_n) = \mathrm{span}(\mathbf{v}_1, \ldots, \mathbf{v}_{i-1}, \mathbf{v}_{i+1}, \ldots, \mathbf{v}_n) \tag{7}$$

by theorem 3 since $\mathbf{0} = 0\mathbf{v}_1 + \cdots + 0\mathbf{v}_{i-1} + 0\mathbf{v}_{i+1} + \cdots + 0\mathbf{v}_n$.

Example 10: Consider the vectors

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{v}_4 = \begin{bmatrix} 2 \\ 2 \\ 2 \\ 2 \end{bmatrix}, \quad \mathbf{v}_5 = \begin{bmatrix} -1 \\ 0 \\ 0 \\ -1 \end{bmatrix}, \tag{8}$$

Show that $\mathrm{span}(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4, \mathbf{v}_5) = \mathrm{span}(\mathbf{v}_1, \mathbf{v}_2)$.

$\mathrm{Span}(\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4, \vec{v}_5) = \mathrm{span}(\vec{v}_1, \vec{v}_2, \vec{v}_4, \vec{v}_5)$ ①

By theorem 3 since $\vec{v}_3 = \vec{v}_1 - \vec{v}_2 + 0\vec{v}_4 + 0\vec{v}_5$

$\mathrm{span}(\vec{v}_1, \vec{v}_2, \vec{v}_4, \vec{v}_5) = \mathrm{span}(\vec{v}_1, \vec{v}_2, \vec{v}_5)$ ②

By theorem 3 since $\vec{v}_4 = 2\vec{v}_1 + 0\vec{v}_2 + 0\vec{v}_5$

$\mathrm{span}(\vec{v}_1, \vec{v}_2, \vec{v}_5) = \mathrm{span}(\vec{v}_1, \vec{v}_2)$ ③

By theorem 3 since $\vec{v}_5 = -\vec{v}_2 + 0\vec{v}_1$

$\mathrm{span}(\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4, \vec{v}_5) = \mathrm{span}(\vec{v}_1, \vec{v}_2)$